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Category and dimensions for cut-out sets ☆

Ying Xiong *, Min Wu

Department of Mathematics, South China University of Technology, Guangzhou 510641, PR China

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ABSTRACT

In this paper, we study the modified box dimensions of cut-out sets that belong to a positive, nonincreasing and summable sequence. Noting that the family of such sets is a compact metric space under the Hausdorff metric, we prove that the lower modified box dimension equals zero and the upper modified box dimension equals the upper box dimension for almost all cut-out set in the sense of Baire category.

1. Introduction and main results

This paper mainly deals with the compact sets of Lebesgue measure zero in Euclidean space. Such sets can be ignored in many situations since there is no essential difference between them and the empty set from the point of measure theory or probability theory. However, they also play an important role in many problems from different areas of mathematics such as number theory, dynamical systems and harmonic analysis. And they are interesting in themselves as theoretical examples and counterexamples. A classical way to study their geometric structure is through the fractal measures and dimensions (see [1–7]).

In some situations, it is convenient to study the complement of such set rather than the set itself. Let A be a convex and compact subset of $\mathbb{R}^d$ and $U_1, U_2, \dots$ be a sequence of disjoint open convex sets contained in A with total Lebesgue measure equal to that of A. We call the set $E = A \setminus \bigcup_i U_i$ a cut-out set. For example, two classical fractal sets: the middle-third Cantor set and Sierpiński gasket are both cut-out set. When dealing with a cut-out set it is a feasible practice to study the position and the shape of the complement component U_i associated with the cut-out set. This idea is very efficient when the cut-out set is a subset of real line since each compact set on the real line of Lebesgue measure zero can be viewed as a cut-out set and each complement component of its must be an open interval. It is therefore not surprising that the papers [8–13] establish relationships between the fractal dimensions of cut-out sets and some properties of the size of their complement intervals.

This paper focuses on the modified box dimensions of cut-out sets on the real line. We start by briefly introducing some definitions and known facts in Section 1.1 and then state our main results in Section 1.2.

1.1. Definitions and some known facts

Definition 1 *(Cut-out set).* (See [8,13].) Let $a = \{a_k\}$ be a positive, nonincreasing and summable sequence. For each closed interval I_a of length $|a| = \sum_{k \geqslant 1} a_k$, we define $\mathscr{C}_a(I_a)$ to be the family of all compact set E contained in I_a with the form

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* Corresponding author.
E-mail addresses: xiongyng@gmail.com (Y. Xiong), wumin_scut@mail.edu.cn (M. Wu).

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$E = I_a \setminus \bigcup_{k \geqslant 1} U_k$, where $\{U_k\}$ is a disjoint family of open intervals contained in I_a such that for all $k \geqslant 1$, U_k is of length $|U_k| = a_k$.

Besicovitch and Taylor [8] investigated the relationships between the Hausdorff dimension of cut-out set $E \in \mathscr{C}_a(I_a)$ and the decay rate of the sequence a. Setting

$$b_n = \frac{r_n}{n}, \quad \text{where } r_n = \sum_{k \geqslant n} a_k,$$

and

$$\alpha(a) = \liminf_{n \to \infty} \alpha_n, \qquad \beta(a) = \limsup_{n \to \infty} \alpha_n, \tag{1.1}$$

where $n b_n^{\alpha_n} = 1$ for all $n \geqslant 1$, among other things, they proved that:

(a) the Hausdorff dimension $\dim_{\mathrm{H}} E \leqslant \alpha(a)$ for all $E \in \mathscr{C}_a(I_a)$;
(b) there exists an $E_\gamma \in \mathscr{C}_a(I_a)$ such that $\dim_{\mathrm{H}} E_\gamma = \gamma$ for all $\gamma \in [0, \alpha(a)]$.

As for the box dimensions of the sets in $\mathscr{C}_a(I_a)$, Falconer in [12, §3.2] showed that every $E \in \mathscr{C}_a(I_a)$ has the same upper box dimension and the same lower box dimension. Moreover, combining the results of Falconer [12, §3.2], Tricot [14] and Garcia et al. [13], it is known that for all $E \in \mathscr{C}_a(I_a)$,

$$\frac{1 - \overline{\dim}_{\mathrm{B}} E}{1 - \underline{\dim}_{\mathrm{B}} E} \cdot \underline{\dim}_{\mathrm{B}} E \leqslant \alpha'(a) \leqslant \underline{\dim}_{\mathrm{B}} E = \alpha(a) \leqslant \overline{\dim}_{\mathrm{B}} E = \beta(a) = \beta'(a), \tag{1.2}$$

where $\alpha(a), \beta(a)$ are defined by (1.1) and

$$\alpha'(a) = \liminf_{k \to \infty} \frac{\log 1/k}{\log a_k}, \qquad \beta'(a) = \limsup_{k \to \infty} \frac{\log 1/k}{\log a_k}. \tag{1.3}$$

In the next subsection, we will state our results about modified box dimensions of cut-out sets. For the definitions of such dimensions, see Falconer's book [15, §3.3].

1.2. Main results

We begin with recalling some terminologies of Baire category. A subset of a complete metric space X is said to be of first category if it can be represented by a countable union of nowhere dense sets. And we say a property holds almost all in the sense of Baire category if it holds except for a set of first category. There is a refined introduction of Baire category and its applications in Oxtoby [16].

It is worth noting that the family of cut-out sets $\mathscr{C}_a(I_a)$ is a compact metric space under the Hausdorff metric ρ (see Proposition 1 in Section 2.3). This is the starting point for our work since we can use the tools of Baire category to describe the modified box dimensions of cut-out sets.

Theorem 1. *Let the sequence a be as in Definition* 1, *$\beta(a)$ defined by* (1.1) *and $\mathscr{C}_a(I_a)$ the family of cut-out sets belonging to the sequence a, then*

$$\dim_{\mathrm{H}} E = \underline{\dim}_{\mathrm{MB}} E = 0 \quad \textit{and} \quad \overline{\dim}_{\mathrm{MB}} E = \beta(a)$$

for almost all $E \in \mathscr{C}_a(I_a)$ in the sense of Baire category.

Remark. In Feng and Wu [17], there is an interesting analogue of Theorem 1. Let $\mathcal{K}^d$ denote the collection of all compact sets in $\mathbb{R}^d$ and ρ denote the Hausdorff metric, then $(\mathcal{K}^d, \rho)$ is a complete metric space. They proved that

$$\dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K = 0 \quad \text{and} \quad \dim_{\mathrm{P}} K = \overline{\dim}_{\mathrm{B}} K = d,$$

for almost all $K \in \mathcal{K}^d$ in the sense of Baire category.

We also investigate the range of modified box dimensions of cut-out set $E \in \mathscr{C}_a(I_a)$ and obtain Theorem 2.

Theorem 2. *Let the sequence a be as in Definition* 1, *$\alpha(a), \beta(a)$ defined by* (1.1) *and $\mathscr{C}_a(I_a)$ the family of cut-out sets belonging to the sequence a. Suppose that $0 \leqslant \underline{d} \leqslant \alpha(a)$ and $0 \leqslant \overline{d} \leqslant \beta(a)$, then the sets*

$$\underline{\mathscr{D}}(\underline{d}) := \{E \in \mathscr{C}_a(I_a)\colon\ \underline{\dim}_{\mathrm{MB}} E = \underline{d}\}, \qquad \overline{\mathscr{D}}(\overline{d}) := \{E \in \mathscr{C}_a(I_a)\colon\ \overline{\dim}_{\mathrm{MB}} E = \overline{d}\}$$

are both dense in the compact metric space $(\mathscr{C}_a(I_a), \rho)$.

This paper is organized as follows. Section 2 is a preliminary including the introduction of the set $C_a(I_a)$, the subfamily $\mathscr{C}_a^n(E)$, the compactness of $\mathscr{C}_a(I_a)$ and Lemmas 1–7. The proofs of the above theorems will be left to Section 3.

2. Preliminaries

In what follows, $a = \{a_k\}$ will always denote a positive, nonincreasing and summable sequence and $\mathscr{C}_a(I_a)$ the family of cut-out sets belonging to a. To avoid ambiguities let us define the complement intervals of a cut-out set.

Definition 2 *(Complement intervals).* Let $E \in \mathscr{C}_a(I_a)$ be a cut-out set. The disjoint open intervals $U_1^E, U_2^E, \ldots$ are called complement intervals of E if

(i) $E = I_a \setminus \bigcup_k U_k^E$;
(ii) U_k^E is of length a_k for all $k \geqslant 1$;
(iii) if the length $|U_k^E| = |U_{k+1}^E|$ for some $j \geqslant 1$, then the open interval U_k^E lies on the left of the interval U_{k+1}^E, i.e.,

$$x < y, \quad \text{if } x \in U_k^E \text{ and } y \in U_{k+1}^E.$$

2.1. The set $C_a(I_a)$ in $\mathscr{C}_a(I_a)$

In this subsection, we introduce a special cut-out set $C_a(I_a) \in \mathscr{C}_a(I_a)$ which appears naturally in the study of $\mathscr{C}_a(I_a)$.

The set $C_a(I_a)$ can be constructed as follows. In the first step, we remove from I_a an open interval of length a_1, then get two closed intervals I_1^1 and I_2^1. Suppose that we get closed intervals $I_1^k, \ldots, I_{2^k}^k$ contained in I_a after the kth step, then we remove from I_j^k an open interval of length a_{2^k+j-1}, obtaining the closed intervals I_{2j-1}^{k+1} and I_{2j}^{k+1}. Finally, we define

$$C_a(I_a) := \bigcap_{k \geqslant 1} \bigcup_{j=1}^{2^k} I_j^k. \tag{2.1}$$

Besicovitch and Taylor [8], Cabrelli et al. [9,10] studied the Hausdorff dimension of $C_a(I_a)$. They showed that

$$\dim_{\mathrm{H}} C_a(I_a) = \underline{\dim}_{\mathrm{B}} C_a(I_a) = \alpha(a), \tag{2.2}$$

where $\alpha(a)$ is defined by (1.1).

As for the modified box dimensions, we have

Lemma 1. *Let $C_a(I_a)$ be the set defined above and α, β defined as in* (1.1)*, then*

$$\underline{\dim}_{\mathrm{MB}} C_a(I_a) = \alpha(a) \quad \textit{and} \quad \overline{\dim}_{\mathrm{MB}} C_a(I_a) = \beta(a).$$

Proof. Since $\dim_{\mathrm{H}} C_a(I_a) \leqslant \underline{\dim}_{\mathrm{MB}} C_a(I_a) \leqslant \underline{\dim}_{\mathrm{B}} C_a(I_a)$, it follows from (2.2) that

$$\underline{\dim}_{\mathrm{MB}} C_a(I_a) = \alpha(a).$$

We now consider the upper modified box dimension. Writing

$$E^{k,j} = C_a(I_a) \cap I_j^k,$$

where I_j^k is as in (2.1), we first show that

$$\overline{\dim}_{\mathrm{B}} E^{k,j} = \beta(a), \quad \text{for all } k \geqslant 1 \text{ and } 1 \leqslant j \leqslant 2^k. \tag{2.3}$$

For two sequences $s = \{s_i\}$ and $s' = \{s'_i\}$, we denote $s \succ s'$ if $s_i \geqslant s'_i$ for all $i \geqslant 1$. Let $a^{k,j} = \{a_i^{k,j}\}$ be the length sequence of complement intervals of $E^{k,j}$. It follows from the construction of $C_a(I_a)$ that

$$a^{k,j} \succ a^{k',j'}, \quad \text{if } k < k' \text{ or } k = k' \text{ and } j < j',$$

since a is nonincreasing. And so (1.2) and (1.3) imply

$$\overline{\dim}_{\mathrm{B}} E^{k,j} \geqslant \overline{\dim}_{\mathrm{B}} E^{k',j'}, \quad \text{if } k < k' \text{ or } k = k' \text{ and } j < j'. \tag{2.4}$$

Notice that for all $k \geqslant 1$,

$$\overline{\dim}_{\mathrm{B}} C_a(I_a) = \max_{1 \leqslant j \leqslant 2^k} \overline{\dim}_{\mathrm{B}} E^{k,j},$$

since $C_a(I_a) = \bigcup_{1 \leqslant j \leqslant 2^k} E^{k,j}$. Therefore, it follows from (2.4) that

$$\beta(a) = \overline{\dim}_B C_a(I_a) \geqslant \overline{\dim}_B E^{k,j} \geqslant \overline{\dim}_B E^{k+1,1} = \max_{1 \leqslant j' \leqslant 2^{k+1}} \overline{\dim}_B E^{k+1,j'} = \overline{\dim}_B C_a(I_a) = \beta(a)$$

for all $k \geqslant 1$ and $1 \leqslant j \leqslant 2^k$. We obtain the equality (2.3).

Now let U be an open interval with $U \cap C_a(I_a) \neq \emptyset$. Then there are $k \geqslant 1$ and $1 \leqslant j \leqslant 2^k$ such that

$$E^{k,j} \subset U \cap C_a(I_a).$$

And so $\overline{\dim}_B U \cap C_a(I_a) = \beta(a)$ according to (2.3). By Proposition 3.6 of [15], we have

$$\overline{\dim}_{MB} C_a(I_a) = \beta(a). \qquad \square$$

Remark. It is well known that the upper modified box dimension coincides with the packing dimension (see [15, §3.4]). And so Lemma 1 can also be derived from Theorem 4.2 of [13]. We gave a proof in order to make the paper self-contained.

2.2. The subfamily $\mathscr{C}_a^n(E)$

Let $E \in \mathscr{C}_a(I_a)$ and $U_1^E, U_2^E, \ldots, U_n^E$ be the first n complement intervals of E for some $n > 1$. Clearly, there is a permutation σ_n^E on the set $\{1, \ldots, n\}$ describing the relative positions of $U_1^E, U_2^E, \ldots, U_n^E$ such that

$$\text{if } x_i \in U^E_{\sigma_n^E(i)} \text{ for } 1 \leqslant i \leqslant n, \text{ then } x_1 < x_2 < \cdots < x_n.$$

We define a subfamily $\mathscr{C}_a^n(E)$ of $\mathscr{C}_a(I_a)$ as

$$\mathscr{C}_a^n(E) := \left\{F \in \mathscr{C}_a(I_a)\colon \text{ if } x_i \in U^F_{\sigma_n^E(i)} \text{ for } 1 \leqslant i \leqslant n, \text{ then } x_1 < x_2 < \cdots < x_n\right\}, \tag{2.5}$$

where $\{U_k^F\}$ are the complement intervals of F defined by Definition 2.

We have the following lemma describing the diameter of $\mathscr{C}_a^n(E)$ with respect to the Hausdorff metric.

Lemma 2. *Let $E \in \mathscr{C}_a(I_a)$ and $\mathscr{C}_a^n(E)$ defined by* (2.5)*, then with respect to the Hausdorff metric,*

$$\operatorname{diam} \mathscr{C}_a^n(E) \leqslant 3r_{n+1},$$

where $r_{n+1} = \sum_{j \geqslant n+1} a_j$.

Proof. Let $l_i(F)$ ($r_i(F)$) be the left endpoint (right endpoint) of the complement interval U_i^F for each $F \in \mathscr{C}_a(I_a)$, and denote

$$F_n = \bigcup_{1 \leqslant i \leqslant n} \left\{l_i(F), r_i(F)\right\}.$$

Suppose that $F, F' \in \mathscr{C}_a^n(E)$, then by the definition of $\mathscr{C}_a^n(E)$, we have

$$\left|l_i(F) - l_i(F')\right| \leqslant r_{n+1} \quad \text{and} \quad \left|r_i(F) - r_i(F')\right| \leqslant r_{n+1}$$

for $1 \leqslant i \leqslant n$. And so $\rho(F_n, F'_n) \leqslant r_{n+1}$. On the other hand, it is plain to see that

$$\rho(F, F_n) \leqslant r_{n+1} \quad \text{and} \quad \rho\left(F', F'_n\right) \leqslant r_{n+1}.$$

Therefore, we have $\rho(F, F') \leqslant 3r_{n+1}$ and the proof is completed. $\square$

2.3. The compactness of $\mathscr{C}_a(I_a)$

In this subsection, we deal with the compactness of the family $\mathscr{C}_a(I_a)$.

Proposition 1. *Let the sequence a be as in Definition* 1*, then the family of cut-out sets $\mathscr{C}_a(I_a)$ belonging to the sequence a is a compact metric space without isolated point with respect to the Hausdorff metric ρ.*

Proof. We first show that the space $(\mathscr{C}_a(I_a), \rho)$ is compact. Let X be a compact metric space and $\mathscr{K}(X)$ be the space of nonempty compact subsets of X with the Hausdorff metric ρ. According to the discussion of [18, §2.10.21], it is known that $(\mathscr{K}(X), \rho)$ is a compact metric space. Since $\mathscr{C}_a(I_a) \subset \mathscr{K}(I_a)$ is a subset of the compact space $(\mathscr{K}(I_a), \rho)$, we only need to show that $\mathscr{C}_a(I_a)$ is closed.

Let $E' \in \overline{\mathscr{C}_a(I_a)}$ with $E' = I_a \setminus \bigcup_j U'_j$, where U'_j is the complement interval of length a'_j (see Definition 2). If we can prove that $a'_j = a_j$ for all $j \geqslant 1$, then $E' \in \mathscr{C}_a(I_a)$ and so $\mathscr{C}_a(I_a)$ is closed.

Let $\varepsilon > 0$, $\delta > 0$ such that $\varepsilon - 2\delta > 0$. Suppose that $E \in \mathscr{C}_a(I_a)$ with $\rho(E, E') < \delta$. Then

$$E \subset E'(\delta) = \{x \in I_a\colon\ d(x, E') \leqslant \delta\}. \tag{2.6}$$

Notice that the open set $I_a \setminus E'(\delta)$ contains at least $\mathrm{card}\{j \geqslant 1\colon\ a'_j \geqslant \varepsilon\}$ disjoint open intervals of length not less than $\varepsilon - 2\delta$. So (2.6) implies

$$\mathrm{card}\{j \geqslant 1\colon\ a_j \geqslant \varepsilon - 2\delta\} \geqslant \mathrm{card}\{j \geqslant 1\colon\ a'_j \geqslant \varepsilon\}.$$

Since δ is arbitrary, we have

$$\mathrm{card}\{j \geqslant 1\colon\ a_j \geqslant \varepsilon\} \geqslant \mathrm{card}\{j \geqslant 1\colon\ a'_j \geqslant \varepsilon\}.$$

It follows that $a_j \geqslant a'_j$ for all $j \geqslant 1$. By the same argument, we also have $a'_j \geqslant a_j$ for all $j \geqslant 1$. Thus we have shown that $E' \in \mathscr{C}_a(I_a)$.

As for the fact that the space $(\mathscr{C}_a(I_a), \rho)$ has no isolated points, we note that for all $E \in \mathscr{C}_a(I_a)$ and $n > 1$, the family $\mathscr{C}_a^n(E)$ contains many members other than E. By Lemma 2, it follows that E is not an isolated point for all $E \in \mathscr{C}_a(I_a)$. □

2.4. Some other lemmas

In this subsection, we present some other lemmas needed in our proof. We begin with the following notation.

Let $J = [l_J, r_J]$ be a closed interval, denote by $E(J)$ the set $E \cap J \cup \{l_J, r_J\}$ for any bounded set E on the real line. Given $\varepsilon > 0$, let

$$V_\varepsilon(E) = \{x \in I_a\colon\ d(x, E) \leqslant \varepsilon\},$$

where $d(x, E) = \inf\{|x - y|\colon\ y \in E\}$. Denote by $N_\delta(E)$ the smallest number of intervals of length δ that cover E.

Lemma 3. *Let $J = [l_J, r_J] \subset I_a$ be a closed interval. Suppose that $E, F \in \mathscr{C}_a(I_a)$ with $\rho(E, F) < \delta$, then*

$$3^{-1} N_\delta(E(J)) \leqslant N_\delta(F(J)) \leqslant 3N_\delta(E(J)).$$

Proof. Since $\rho(E, F) < \delta$, we have

$$F(J) \subset V_\delta(E(J)).$$

Together with the fact that $N_\delta(V_\delta(K)) \leqslant 3N_\delta(K)$ for all bounded set K on the real line, we have

$$N_\delta(F(J)) \leqslant 3N_\delta(E(J)).$$

The other inequality can be obtained by the same argument. □

Lemma 4. *Let $J = [l_J, r_J] \subset I_a$ be a closed interval, then the family*

$$\mathscr{F}_J := \{E \in \mathscr{C}_a(I_a)\colon\ \underline{\dim}_B E \cap J = 0\}$$

is a dense G_δ subset of $\mathscr{C}_a(I_a)$ under the Hausdorff metric ρ for all $J \neq I_a$.

Proof. Suppose that $I_a = [l, r]$. We first show that $\mathscr{F}_J$ is dense. Since $J \neq I_a$, there are two cases to be considered.

Case 1. $J \subset [l, r - \xi]$ for some $\xi > 0$;
Case 2. $J \subset [l - \xi, r]$ for some $\xi > 0$.

We now consider Case 1. For any $E \in \mathscr{C}_a(I_a)$ and $n > 1$, let σ_n^E be the permutation as in Section 2.2. Put

$$s_k^n(E) = \begin{cases} \sum_{i=1}^k a_{\sigma_n^E(i)}, & \text{for } 1 \leqslant k \leqslant n, \\ \sum_{i=1}^k a_i, & \text{for } k > n, \end{cases}$$

and

$$E_n = \{l, r\} \cup \bigcup_{k \geqslant 1} \{l + s_k^n(E)\}.$$

Clearly, $E_n \in \mathscr{C}_a^n(E)$, which follows $\rho(E, E_n) \to 0$ due to Lemma 2. On the other hand, observe that $E_n \in \mathscr{F}_J$ since $E_n \cap J \subset E_n \cap [l, r-\xi]$ is finite. Consequently, since E was arbitrary, we see that $\mathscr{F}_J$ is dense. The proof of Case 2 is similar.

Next, we will construct a dense G_δ set $\mathscr{W}$ from $\mathscr{F}_J$. For each $E \in \mathscr{F}_J$, there is a sequence of real number $\{\delta_j(E)\}_{j\geqslant 1}$ such that

$$0 < \delta_j(E) < 2^{-j} \quad \text{and} \quad \frac{\log N_{\delta_j(E)}(E(J))}{-\log \delta_j(E)} < 2^{-j}, \quad \text{for all } j \geqslant 1, \tag{2.7}$$

due to the fact $\underline{\dim}_{\mathrm{B}}\, J \cap E = 0$. Then, for each $j \geqslant 1$, define

$$\mathscr{W}_j = \bigcup_{E \in \mathscr{F}_J} B\big(E, \delta_j(E)\big),$$

where $B(E, r) = \{K \in \mathscr{C}_a(I_a)\colon\ \rho(E, K) < r\}$. Clearly, $\mathscr{W}_j$ is an open and dense subset of $\mathscr{C}_a(I_a)$ since $\mathscr{F}_J \subset \mathscr{W}_j$. Now let $\mathscr{W} = \bigcap_j \mathscr{W}_j$, it follows that $\mathscr{W}$ is a dense G_δ set.

Finally, if we show that $\mathscr{F}_J = \mathscr{W}$ then the proof is completed. It is clear that $\mathscr{F}_J \subset \mathscr{W}$, so we only need to prove $\mathscr{W} \subset \mathscr{F}_J$. Let $K \in \mathscr{W}$, then $K \in \mathscr{W}_j$ for all $j \geqslant 1$. By the definition of $\mathscr{W}_j$, there exists $K_j \in \mathscr{F}_J$ such that $\rho(K, K_j) < \delta_j(K_j)$. By Lemma 3 and (2.7), we have

$$\underline{\dim}_{\mathrm{B}}\, K(J) \leqslant \liminf_{j\to\infty} \frac{\log N_{\delta_j(K_j)}(K(J))}{-\log \delta_j(K_j)} = \liminf_{j\to\infty} \frac{\log N_{\delta_j(K_j)}(K_j(J))}{-\log \delta_j(K_j)} = 0.$$

It follows that $\underline{\dim}_{\mathrm{B}}\, J \cap K = 0$, and so $K \in \mathscr{F}_J$. $\square$

Lemma 5. *Let $J = [l_J, r_J] \subset I_a$ be a closed interval and $\beta(a)$ defined by* (1.1), *then the family*

$$\mathscr{G}_J := \big\{E \in \mathscr{C}_a(I_a)\colon\ \overline{\dim}_{\mathrm{B}}\, E \cap J = \beta(a) \text{ or } E \cap J = \emptyset\big\}$$

is a dense G_δ subset of $\mathscr{C}_a(I_a)$ under the Hausdorff metric ρ for all J.

Proof. Suppose that $I_a = [l, r]$. We first show that $\mathscr{G}_J$ is dense. For this, suppose that $E \in \mathscr{C}_a(I_a)$ with $E \cap J \neq \emptyset$ and $\overline{\dim}_{\mathrm{B}}\, E \cap J < \beta(a)$. Thus we have

$$\overline{\dim}_{\mathrm{B}}\, E \cap [l, l_J] = \beta(a) \tag{2.8}$$

or

$$\overline{\dim}_{\mathrm{B}}\, E \cap [r_J, r] = \beta(a). \tag{2.9}$$

If we can prove $E \in \overline{\mathscr{G}_J}$ when (2.8) holds, then by the symmetry, we also have $E \in \overline{\mathscr{G}_J}$ when (2.9) holds. And it follows that $\mathscr{G}_J$ is dense. There are two cases to be consider.

Case 1. $E \cap (l_J, r_J] \neq \emptyset$;
Case 2. $E \cap (l_J, r_J] = \emptyset$.

For Case 1, suppose that $p \in E \cap (l_J, r_J]$ and set $\Delta = p - l_J$. For any $0 < \varepsilon < \Delta$, there are closed intervals $J_1, \dots, J_n$ of length less than ε such that $[l, l_J] = J_1 \cup \cdots \cup J_n$. Clearly, we can find $1 \leqslant k \leqslant n$ such that $\overline{\dim}_{\mathrm{B}}\, E \cap J_k = \beta(a)$ due to (2.8). Put

$$l_\varepsilon = \min\{x\colon\ x \in E \cap J_k\} \quad \text{and} \quad r_\varepsilon = \max\{x\colon\ x \in E \cap J_k\}.$$

The discussion above gives

$$\overline{\dim}_{\mathrm{B}}\, E \cap [l_\varepsilon, r_\varepsilon] = \beta(a) \quad \text{and} \quad r_\varepsilon - l_\varepsilon < \varepsilon. \tag{2.10}$$

Now define $E_\varepsilon = K_1 \cup K_2 \cup K_3 \cup K_4$, where

$$K_1 = E \cap [l, l_\varepsilon], \qquad K_2 = \big\{x + l_\varepsilon - r_\varepsilon\colon\ x \in E \cap [r_\varepsilon, p]\big\},$$
$$K_3 = \big\{x + p - r_\varepsilon\colon\ x \in E \cap [l_\varepsilon, r_\varepsilon]\big\}, \qquad K_4 = E \cap [p, r].$$

It is not difficult to verify that $E_\varepsilon \in \mathscr{C}_a(I_a)$. Furthermore, a simple calculation shows that $K_3 \subset J$, and so

$$\overline{\dim}_{\mathrm{B}}\, E_\varepsilon \cap J \geqslant \overline{\dim}_{\mathrm{B}}\, K_3 = \beta(a),$$

hence $E_\varepsilon \in \mathscr{G}_J$. Moreover, noticing that $\rho(E, E_\varepsilon) < \varepsilon$ since $r_\varepsilon - l_\varepsilon < \varepsilon$, considering that ε was arbitrary, we have proven $E \in \overline{\mathscr{G}_J}$ in the Case 1.

For Case 2, let $p = \min\{x \in E:\ x > r_J\}$ (note that $r \in E$) and set $\Delta = p - r_J$. For any $0 < \varepsilon < \Delta$, there are $l_\varepsilon, r_\varepsilon \in E \cap [l, l_J]$ such that $0 < r_\varepsilon - l_\varepsilon < \varepsilon$ due to $\overline{\dim}_B E \cap [l, l_J] = \beta(a) > 0$. Then define E_ε as in Case 1. A simple calculation shows that $E_\varepsilon \cap J = \emptyset$. Together with the fact that $\rho(E, E_\varepsilon) < \varepsilon$, we see that $E \in \overline{\mathscr{G}_J}$ also holds in Case 2.

Therefore, in both cases above, we have proved $E \in \overline{\mathscr{G}_J}$ if (2.8) holds, and so $\mathscr{G}_J$ is dense.

Next, we will construct a dense G_δ set $\mathscr{W}$ from $\mathscr{G}_J$. For this, write

$$\mathscr{G}_J^\beta = \{E \in \mathscr{G}_J:\ \overline{\dim}_B E \cap J = \beta(a)\}, \qquad \mathscr{G}_J^0 = \{F \in \mathscr{G}_J:\ F \cap J = \emptyset\}.$$

For each $E \in \mathscr{G}_J^\beta$, there is a sequence of real number $\{\delta_j(E)\}_{j \geqslant 1}$ such that

$$0 < \delta_j(E) < 2^{-j} \quad \text{and} \quad \left| \frac{\log N_{\delta_j(E)}(E(J))}{-\log \delta_j(E)} - \beta(a) \right| < 2^{-j}, \quad \text{for all } j \geqslant 1, \tag{2.11}$$

due to $\overline{\dim}_B E \cap J = \beta(a)$. Now, for each $F \in \mathscr{G}_J^0$, there is a $\delta(F) > 0$ such that $K \cap J = \emptyset$ for all $K \in \mathscr{C}_a(I_a)$ with $\rho(K, F) < \delta(F)$ due to $F \cap J = \emptyset$. Then, for each $j \geqslant 1$, define

$$\mathscr{V}_j^\beta = \bigcup_{E \in \mathscr{G}_J^\beta} B(E, \delta_j(E)), \qquad \mathscr{V}^0 = \bigcup_{F \in \mathscr{G}_J^0} B(F, \delta(F)),$$

where $B(E, r) = \{K \in \mathscr{C}_a(I_a):\ \rho(E, K) < r\}$. Clearly, $\mathscr{V}_j^\beta \cup \mathscr{V}^0$ is an open and dense subset of $\mathscr{C}_a(I_a)$ since $\mathscr{G}_J \subset \mathscr{V}_j^\beta \cup \mathscr{V}_j^0$. Now let $\mathscr{V} = \mathscr{V}^0 \cup \bigcap_j \mathscr{V}_j^\beta$, it follows that $\mathscr{V}$ is a dense G_δ set.

Finally, if we show that $\mathscr{G}_J = \mathscr{V}$ then the proof is completed. It is clear that $\mathscr{V}^0 \subset \mathscr{G}_J \subset \mathscr{V}$, so we only need to prove $\bigcap_j \mathscr{V}_j^\beta \subset \mathscr{G}_J$. Let $K \in \bigcap_j \mathscr{V}_j^\beta$, by the definition of $\mathscr{V}_j^\beta$, there exists $K_j \in \mathscr{G}_J^\beta$ such that $\rho(K, K_j) < \delta_j(K_j)$. By Lemma 3 and (2.11), we have

$$\overline{\dim}_B K(J) \geqslant \limsup_{j \to \infty} \frac{\log N_{\delta_j(K_j)}(K(J))}{-\log \delta_j(K_j)} = \limsup_{j \to \infty} \frac{\log N_{\delta_j(K_j)}(K_j(J))}{-\log \delta_j(K_j)} = \beta(a).$$

It follows that $\overline{\dim}_B J \cap K = \beta(a)$, and so $K \in \mathscr{G}_J$. $\square$

Lemma 6. *Let $a = \{a_k\}$ be a positive, nonincreasing and summable sequence and $\alpha(a)$ defined by* (1.1)*. Then for all $0 < \underline{d} < \alpha(a)$, there is a subsequence $\underline{a}$ of a such that*

$$\alpha(\underline{a}) = \underline{d}.$$

Proof. For each sequence b, write

$$\alpha_n(b) = \left(1 - \frac{\log \sum_{i \geqslant n} b_i}{\log n}\right)^{-1}.$$

By (1.1), we have

$$\alpha(a) = \liminf_{n \to \infty} \alpha_n(a).$$

Before constructing $\underline{a}$, we first define $\{a^j\}_{j \geqslant 0}$ by induction, where each a^j is a sequence constructed by removing finite items from the sequence a.

Let $a^0 = a$. If $\{a^i\}_{i \leqslant j}$ are determined, we define a^{j+1} as follows.

Notice that there is an $n_j > 2^j$ such that $\alpha_k(a^j) > \underline{d}$ for all $k > n_j$, since $\alpha(a^j) = \alpha(a) > \underline{d}$. Now choose $m_j > n_j$ large enough such that for all $k \leqslant n_j$,

$$\begin{aligned} &\left(1 - \frac{\log \sum_{m_j \geqslant i \geqslant k} a_i^j}{\log k}\right)^{-1} > \underline{d}, \quad \text{if } \alpha_k(a^j) > \underline{d}, \\ &\alpha_k(a^j) - \left(1 - \frac{\log \sum_{m_j \geqslant i \geqslant k} a_i^j}{\log k}\right)^{-1} < 2^{-j}, \quad \text{if } \alpha_k(a^j) \leqslant \underline{d}. \end{aligned} \tag{2.12}$$

Then let $l_j > m_j$ be the smallest integer satisfying one of the following two conditions.

Condition 1. There exists an integer $k \in [n_j + 1, m_j]$ such that

$$\left(1 - \frac{\log(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j} a_i^j)}{\log k}\right)^{-1} \leqslant \underline{d}.$$

Condition 2. There exists an integer $k \geqslant l_j$ such that

$$\left(1 - \frac{\log \sum_{i \geqslant k} a_i^j}{\log(k - l_j + m_j + 1)}\right)^{-1} \leqslant \underline{d}.$$

Finally, we define

$$a_i^{j+1} = \begin{cases} a_i^j, & \text{if } i \leqslant m_j, \\ a_{i-m_j-1+l_j}^j, & \text{if } i > m_j. \end{cases}$$

Now observe that if k is such as in Condition 1, then

$$\alpha_k(a^{j+1}) = \left(1 - \frac{\log(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j} a_i^j)}{\log k}\right)^{-1} \leqslant \underline{d} < \left(1 - \frac{\log(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j - 1} a_i^j)}{\log k}\right)^{-1},$$

since $l_j - 1$ does not satisfy Condition 1. And so

$$\underline{d} - \alpha_k(a^{j+1}) < \left(1 - \frac{\log(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j - 1} a_i^j)}{\log k}\right)^{-1} - \left(1 - \frac{\log(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j} a_i^j)}{\log k}\right)^{-1}.$$

Noticing that

$$\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j - 1} a_i^j < 2\left(\sum_{m_j \geqslant i \geqslant k} a_i^j + \sum_{i \geqslant l_j} a_i^j\right),$$

we have

$$\underline{d} - \alpha_k(a^{j+1}) < \log 2 / \log k < j^{-1}, \tag{2.13}$$

due to $k > n^j > 2^j$. While if k is such as in Condition 2, then

$$\alpha_k(a^{j+1}) = \left(1 - \frac{\log \sum_{i \geqslant k} a_i^j}{\log(k - l_j + m_j + 1)}\right)^{-1} \leqslant \underline{d} < \left(1 - \frac{\log \sum_{i \geqslant k} a_i^j}{\log(k - (l_j - 1) + m_j + 1)}\right)^{-1}, \tag{2.14}$$

since $l_j - 1$ does not satisfy Condition 2. And so

$$\begin{aligned} \underline{d} - \alpha_k(a^{j+1}) &< \left(1 - \frac{\log \sum_{i \geqslant k} a_i^j}{\log(k - l_j + m_j + 2)}\right)^{-1} - \left(1 - \frac{\log \sum_{i \geqslant k} a_i^j}{\log(k - l_j + m_j + 1)}\right)^{-1} \\ &\leqslant \frac{-\log \sum_{i \geqslant k} a_i^j}{\log(k - l_j + m_j + 2)} \cdot \frac{\log(k - l_j + m_j + 2) - \log(k - l_j + m_j + 1)}{\log(k - l_j + m_j + 1)} \\ &< (\underline{d}^{-1} - 1) \cdot \frac{1}{(k - l_j + m_j + 1) \log(k - l_j + m_j + 1)} \quad \bigl(\text{by (2.14)}\bigr) \\ &< 2^{-j}(\underline{d}^{-1} - 1) \quad \bigl(\text{since } k \geqslant l_j \text{ and } m_j > n_j > 2^j\bigr). \end{aligned}$$

Together with (2.13), we see that if $k > n_j$ and $\alpha_k(a^{j+1}) \leqslant \underline{d}$, then

$$0 \leqslant \underline{d} - \alpha_k(a^{j+1}) < j^{-1} + 2^{-j}(\underline{d}^{-1} - 1). \tag{2.15}$$

Now we define $\underline{a}$ as

$$\underline{a}_i = a_i^j, \quad \text{if } i \leqslant n_j.$$

Due to the construction of a^j, it is not difficulty to see that for all $j > 1$, there exists at least one integer $k \in [n_j + 1, n_{j+1}]$ with $\alpha_k(\underline{a}) \leqslant \underline{d}$. And for all such k, we have

$$0 \leqslant \underline{d} - \alpha_k(\underline{a}) < j^{-1} + 2^{-j}\big(\underline{d}^{-1} - 1\big) + \sum_{i \geqslant j+1} 2^{-i} = j^{-1} + 2^{-j}\underline{d}^{-1}$$

by (2.12) and (2.15). Therefore, we have $\alpha(\underline{a}) = \underline{d}$. $\square$

Lemma 7. *Let $a = \{a_k\}$ be a positive, nonincreasing and summable sequence and $\beta(a)$ defined by (1.1). Then for all $0 < \bar{d} < \beta(a)$, there is a subsequence $\bar{a}$ of a such that*

$$\beta(\bar{a}) = \bar{d}.$$

Proof. By (1.2) and (1.3), we need to construct a subsequence $\bar{a}$ such that

$$\beta(\bar{a}) = \limsup_{m\to\infty} \frac{-\log m}{\log \bar{a}_m} = \bar{d}.$$

We will do it by induction.

Let $\bar{a}_1 = a_1$. If $\{\bar{a}_i\}_{i\leqslant j}$ are determined as $\bar{a}_i = a_{m_i}$ for $1 \leqslant i \leqslant j$, for the definition of $\bar{a}_{j+1}$ we consider two cases according to the value of $-\log(j+1)/\log a_{m_j+1}$.

(i) $-\log(j+1)/\log a_{m_j+1} \leqslant \bar{d}$, then we define $\bar{a}_{j+1} = a_{m_j+1}$, i.e., $m_{j+1} = m_j + 1$.
(ii) $-\log(j+1)/\log a_{m_j+1} > \bar{d}$, then we define $\bar{a}_{j+1} = a_{m_{j+1}}$ where $m_{j+1} > m_j$ is the smallest number such that $-\log(j+1)/\log a_{m_{j+1}} \leqslant \bar{d}$.

Clearly, we have

$$\beta(\bar{a}) = \limsup_{m\to\infty} \frac{-\log m}{\log \bar{a}_m} \leqslant \bar{d}.$$

On the other hand, notice that the inequality $m_{j+1} > m_j$ must hold for infinitely many $j > 1$ due to $\beta(a) > \bar{d}$. And for such j, we have

$$\frac{-\log j}{\log \bar{a}_j} = \frac{-\log j}{\log a_{m_j}} \leqslant \bar{d} < \frac{-\log(j+1)}{\log a_{m_j+1}}.$$

Together with the fact that

$$\frac{-\log(j+1)}{\log a_{m_j+1}} - \frac{-\log j}{\log a_{m_j}} \leqslant \frac{\log(j+1) - \log(j)}{-\log a_{m_j}} \to 0$$

as $j \to \infty$, we have $\beta(\bar{a}) \geqslant \bar{d}$. Therefore, the sequence $\bar{a}$ is as desired. $\square$

3. Proofs of main results

3.1. Proof of Theorem 1

Suppose that $I_a = [l, r]$. We first deal with the case of lower modified box dimension. Let

$$J_1 = \big[l, (l+r)/2\big], \qquad J_2 = \big[(l+r)/2, r\big].$$

According to Lemma 4, we know that the family $\mathscr{F}_{J_1} \cap \mathscr{F}_{J_2}$ is a dense G_δ set of $\mathscr{C}_a(I_a)$, and so its complement is a set of first category. We will prove that $\underline{\dim}_{\mathrm{MB}} E = 0$ for all $E \in \mathscr{F}_{J_1} \cap \mathscr{F}_{J_2}$, which implies that

$$\dim_{\mathrm{H}} E = \underline{\dim}_{\mathrm{MB}} E = 0, \quad \text{for almost all } E \in \mathscr{C}_a(I_a).$$

In fact, if $E \in \mathscr{F}_{J_1} \cap \mathscr{F}_{J_2}$, by the definition of $\mathscr{F}_J$, we have

$$\underline{\dim}_{\mathrm{B}} E \cap J_1 = 0 \quad \text{and} \quad \underline{\dim}_{\mathrm{B}} E \cap J_2 = 0.$$

Hence

$$\underline{\dim}_{\mathrm{MB}} E = \underline{\dim}_{\mathrm{MB}} E \cap I_a = \underline{\dim}_{\mathrm{MB}} (E \cap J_1) \cup (E \cap J_2) = 0.$$

Now we take up the case of upper modified box dimension. Let

$$\mathscr{J} = \big\{J \subset I_a\colon\ J = [l_J, r_J] \text{ with } l_J, r_J \in \mathbb{Q} \text{ and } l_J < r_J\big\}.$$

According to Lemma 5, the family $\bigcap_{J\in\mathscr{J}}\mathscr{G}_J$ is a dense G_δ set of $\mathscr{C}_a(I_a)$ since $\mathscr{J}$ is countable. We will prove that $\overline{\dim}_{\mathrm{MB}} E=\beta(a)$ for all $E\in\bigcap_{J\in\mathscr{J}}\mathscr{G}_J$, which implies that

$$\overline{\dim}_{\mathrm{MB}} E=\beta(a),\quad \text{for almost all } E\in\mathscr{C}_a(I_a).$$

Suppose that $E\in\bigcap_{J\in\mathscr{J}}\mathscr{G}_J$. By Proposition 3.6 of [15], if we can show that

$$\overline{\dim}_{\mathrm{B}} E\cap V=\overline{\dim}_{\mathrm{B}} E=\beta(a)$$

for all open set V that intersect E, then we have $\overline{\dim}_{\mathrm{MB}} E=\beta(a)$. In fact, if $E\cap V\neq\emptyset$, then there is a $J\in\mathscr{J}$ such that $J\subset V$ and $E\cap J\neq\emptyset$. Since $E\in\mathscr{G}_J$, by the definition of $\mathscr{G}_J$, we have

$$\beta(a)\geqslant\overline{\dim}_{\mathrm{B}} E\cap V\geqslant\overline{\dim}_{\mathrm{B}} E\cap J=\beta(a).$$

Therefore, $\overline{\dim}_{\mathrm{MB}} E=\beta(a)$ and the proof is completed.

3.2. Proof of Theorem 2

Suppose that $I_a=[l,r]$. We first deal with the case of lower modified box dimension. Let $E\in\mathscr{C}_a(I_a)$ and σ_n^E be the permutation on $\{1,\dots,n\}$ as in Section 2.2. By Lemma 2, if we can show that $\underline{\mathscr{D}}(\underline{d})\cap\mathscr{C}_a^n(E)\neq\emptyset$ for all $n>1$, then the set $\underline{\mathscr{D}}(\underline{d})$ is dense.

There are three cases to be considered.

Case 1. $\underline{d}=0$. For each $n>1$, put

$$s_k^n(E)=\begin{cases}\sum_{i=1}^k a_{\sigma_n^E(i)}, & \text{for } 1\leqslant k\leqslant n,\\ \sum_{i=1}^k a_i, & \text{for } k>n,\end{cases}$$

and

$$E_n=\{l,r\}\cup\bigcup_{k\geqslant 1}\bigl\{l+s_k^n(E)\bigr\}.$$

Clearly, $E_n\in\mathscr{C}_a^n(E)$. And $E_n\in\underline{\mathscr{D}}(0)$ since E_n is countable.

Case 2. $\underline{d}=\alpha(a)$. For each $n>1$, put

$$s_k^n(E)=\sum_{i=1}^k a_{\sigma_n^E(i)},\quad \text{for } 1\leqslant k\leqslant n,$$

and

$$E_n=\{l\}\cup\bigcup_{k=1}^n\bigl\{l+s_k^n(E)\bigr\}.$$

Define a sequence a^n as $a_i^n=a_{n+i}$ for all $i\geqslant 1$ and a closed interval

$$I_n=\bigl[l+s_n^n(E),r\bigr].$$

Clearly, $E_n\cup C_{a^n}(I_n)\in\mathscr{C}_a^n(E)$, where $C_{a^n}(I_n)$ is defined in Section 2.1. And according to Lemma 1, we have

$$\underline{\dim}_{\mathrm{MB}} E_n\cup C_{a^n}(I_n)=\underline{\dim}_{\mathrm{MB}} C_{a^n}(I_n)=\alpha\bigl(a^n\bigr)=\alpha(a).$$

Therefore, $E_n\cup C_{a^n}(I_n)\in\mathscr{C}_a^n(E)\cap\underline{\mathscr{D}}(\alpha(a))$.

Case 3. $0<\underline{d}<\alpha(a)$. For each $n>1$, put

$$s_k^n(E)=\sum_{i=1}^k a_{\sigma_n^E(i)},\quad \text{for } 1\leqslant k\leqslant n,$$

and

$$E_n=\{l\}\cup\bigcup_{k=1}^n\bigl\{l+s_k^n(E)\bigr\}.$$

Define a sequence a^n as $a_i^n = a_{n+i}$ for all $i \geqslant 1$. By Lemma 6, there is a subsequence $\underline{a}^n$ of a^n such that $\alpha(\underline{a}^n) = \underline{d}$. Let b^n be the nonincreasing sequence constructed from a^n by removing all items of $\underline{a}^n$. Define

$$F_n = \left\{ l + s_n^n(E) + \sum_{i=1}^{k} b_i^n \colon k \geqslant 1 \right\},$$

and $I_n = [l + s_n^n(E) + \sum_{i \geqslant 1} b_i^n, r]$.

Clearly, $E_n \cup F_n \cup C_{\underline{a}^n}(I_n) \in \mathscr{C}_a^n(E)$, where $C_{\underline{a}^n}(I_n)$ is defined in Section 2.1. Since the set $E_n \cup F_n$ is countable, together with Lemma 2, we have

$$\underline{\dim}_{\mathrm{MB}}\, E_n \cup F_n \cup C_{\underline{a}^n}(I_n) = \underline{\dim}_{\mathrm{MB}}\, C_{\underline{a}^n}(I_n) = \alpha\big(\underline{a}^n\big) = \underline{d}.$$

Therefore, $E_n \cup F_n \cup C_{\underline{a}^n}(I_n) \in \mathscr{C}_a^n(E) \cap \underline{\mathscr{D}}(\underline{d})$.

Summing up all the cases above, we have proved the set $\underline{\mathscr{D}}(\underline{d})$ is dense for all $0 \leqslant \underline{d} \leqslant \alpha(a)$.

As for the cases of upper modified box dimension, by the same argument above only except for replacing Lemma 6 by Lemma 7, we can also prove that the set $\overline{\mathscr{D}}(\overline{d})$ is dense for all $0 \leqslant \overline{d} \leqslant \beta(a)$.

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